The shear dynamics in Bianchi I cosmological model on the brane

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Abstract

The shear dynamics in Bianchi I cosmological model on the brane with a perfect fluid (the equation of state is $p = (\gamma - 1)\mu$) is studied. It is shown that for $1 < \gamma < 2$ the shear parameter has maximum at some moment during a transition period from nonstandard to standard cosmology. An exact formula for the matter density μ in the epoch of maximum shear parameter as a function of the equation of state is obtained.

In recent several years a lot of effort has been done on the idea that our Universe is a boundary of a space-time manifold with a larger number of dimensions [1]. The cosmological evolution of a three-brane in five dimensional space-time become the matter of intense investigations. The corrections to Einstein equations on the brane arising from the influence of a bulk geometry on the brane were obtained [2]. The corresponding cosmological dynamics was studied in detail for a FRW Universe. It was shown that the dynamics of the early brane Universe can differ significantly from the standard scenario [3]. On the other hand, the standard FRW cosmology for the late Universe (when the matter density on the brane μ become negligible in comparison with the brane tension λ) was recovered [4].

Later on this analisys has been done for an anisotropic Bianchi I metric on the brane. The matter source was chosen in the form of a massive scalar field [5] or perfect fluid with the equation of state $p = (\gamma - 1)\mu$ [6] (see also [7] where some exact solutions were found). One of the most interesting results obtained in this way is that the initial singularity, being anisotropic in ordinary cosmology (with the exception of the maximally stiff fluid case [8]), becomes isotropic in the brane cosmology for $\gamma \in (1, 2)$[5, 6]. Since during the evolution of the brane Universe its dynamics approaches the standard one, later stages of the Universe are isotropic for $\gamma < 2$. So, the intermediate stage of a relatively high anisotropy can exist if the matter filling the Universe has a positive

pressure. Intuitively we could expect that the anisotropy maximum is reached sometimes in the transitional period when the matter density is comparable with the brane tension. In this paper we calculate the ratio μ/λ during the maximum anisotropy epoch. We will see that for the radiation-dominated Universe the above suggestion appears to be true, though for an arbitrary γ from the interval $(1, 2)$ (we study only this type of matter in this paper) the epoch of maximal anisotropy can differ significantly from the period when $\mu \sim \lambda$. To conclude this introductory part we remind that current experimental limit on the brane tension is $\lambda > (100\mathrm{GeV})^4$ [9].

The field equations on the brane are [9]

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \tilde{\kappa}^4 S_{\mu\nu} - \mathcal{E}_{\mu\nu}. \tag{1}$$

Here $\tilde{\kappa}$ is the 5-dimensional gravitational constant, κ is the effective 4-dimensional gravitation constant on the brane. Below we assume that the effective cosmological constant on the brane $\Lambda = 0$. Bulk corrections to the Einstein equations on the brane are of two forms: there are quadratic energy-momentum corrections via the tensor $S_{\mu\nu}$ and nonlocal effects from the free gravitational field in the bulk. The energy-momentum corrections are given by

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_\mu{}^\delta T_{\mu\delta} + \frac{1}{24} g_{\mu\nu} \left[3T^{\delta\rho} T_{\delta\rho} - T^2 \right], \tag{2}$$

where $T \equiv T_\mu{}^\mu$. If we define u^μ as the 4-velocity comoving with matter, the nonlocal term has the form

$$\mathcal{E}_{\mu\nu} = \frac{-6}{\kappa^2 \lambda} \left[\mathcal{U} \left(u_\mu u_\nu + \tfrac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_\mu u_\nu + \mathcal{Q}_\nu u_\mu \right], \tag{3}$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. Here

$$\mathcal{U} = -\frac{1}{6} \kappa^2 \lambda \, \mathcal{E}_{\mu\nu} u^\mu u^\nu$$

is an effective nonlocal energy on the brane,

$$\mathcal{P}_{\mu\nu} = -\frac{1}{6} \kappa^2 \lambda \left[h_\mu{}^\alpha h_\nu{}^\beta - \frac{1}{3} h^{\alpha\beta} h_{\mu\nu} \right] \mathcal{E}_{\alpha\beta}$$

is an effective nonlocal anisotropic stress and

$$\mathcal{Q}_\mu = \frac{1}{6} \kappa^2 \lambda \, h_\mu{}^\alpha \mathcal{E}_{\alpha\beta} u^\beta$$

is an effective nonlocal energy flux.

There are also conservation equations for the brane energy-momentum tensor $\bigtriangledown^\mu T_{\mu\nu} = 0$. Full description of dynamical and conservation equations see in [9].

We will study Bianchi I model on the brane with the metric

$$ds^2 = -dt^2 + a_i^2(t) (dx^i)^2, \tag{4}$$

The conservation equations in this case reduce to

$$\dot{\mu} + \Theta(\mu + p) = 0, \tag{5}$$

$$\dot{\mathcal{U}} + \frac{4}{3}\Theta\mathcal{U} + \sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0, \tag{6}$$

$$D^{\nu}\mathcal{P}_{\mu\nu} = 0, \tag{7}$$

where Θ is the volume expansion rate, and $\sigma_{\mu\nu}$ is the shear. The nonlocal energy flux $\mathcal{Q}$ vanishes in this case identically and we have no evolution equation for $\mathcal{P}_{\mu\nu}$ which is a bulk degree of freedom and can not be predicted from the brane [5, 9]. In terms of the mean scale factor $a = (a_1 a_2 a_3)^{1/3}$ the expansion rate $\Theta = 3H = 3\dot{a}/a$.

The Raychaudhuri equation on the brane is [5]

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \sigma^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{2}\kappa^2(\mu + 3p) = -\frac{1}{2}\kappa^2(2\mu + 3p)\frac{\mu}{\lambda} - \frac{6\mathcal{U}}{\kappa^2\lambda} \tag{8}$$

and Gauss-Codazzi equations are [5]

$$\dot{\sigma}_{\mu\nu} + \Theta\sigma_{\mu\nu} = \frac{6}{\kappa^2\lambda}\mathcal{P}_{\mu\nu}, \tag{9}$$

$$-\frac{2}{3}\Theta^2 + \sigma^{\mu\nu}\sigma_{\mu\nu} + 2\kappa^2\mu = -\kappa^2\frac{\mu^2}{\lambda} - \frac{12\mathcal{U}}{\kappa^2\lambda}. \tag{10}$$

Due to the presence of the nonlocal stress $\mathcal{P}_{\mu\nu}$ this system of equations is not closed. In previous papers [5, 6] this difficulty was avoided by choosing $\mathcal{U} = 0$. This condition, however, is sufficient, but not necessary for closing the system. The full situation can be described as follows.

The most restrictive condition is $\mathcal{P}_{\mu\nu} = 0$. In this case it is possible to solve the equation for the shear *separately* for components of the shear tensor. The less restrictive condition $\sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0$ allows us to find the dynamics of shear scalar $\sigma^{\mu\nu}\sigma_{\mu\nu}$ leaving components of the shear tensor undefined.

Taking into account Eq. (6), we can see that both these assumptions lead to Friedmann-like behavior of the "dark radiation" term

$$\mathcal{U} = \frac{C}{a^4}$$

with $C = Const.$

The condition $\mathcal{U} = 0$ automatically leads to $\sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0$ (see Eq. (6)). However, the latter condition is more general. We assume further that $\sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0$ is satisfied. In the opposite case the nonlocal anisotropic stress changes significantly the dynamical equations for shear and "dark radiation". Though even in this case there are some possibilities for closing the system (8)-(10) [10] the resulting dynamics can be very different from studied in this paper and should be specially investigated.

It should be noted that though there are no restriction to choose $\mathcal{P}_{\mu\nu}$ on the brane, we do not know whether a particular chose is consistent in a bulk. The general form of bulk anisotropic metric is not known yet (the first approach to this problem see in [11]). We leave the problem to check that $\mathcal{P}_{\mu\nu}$ chosen on the brane is consistent with the full 5-dimensional metric to future investigations.

It is convenient to introduce the deceleration parameter q as

$$q = -\frac{\ddot{a}a}{\dot{a}^2},$$

or, equivalently,

$$\dot{H} = -(1+q)H^2. \tag{11}$$

This equation in combination with Eq.(8), which can be written in the form

$$\dot{H} = -H^2 - \frac{2}{3}\sigma^2 - \frac{\kappa^2\mu}{6}(3\gamma-2) - \frac{\kappa^2\mu^2}{6\lambda}(3\gamma-1) - \frac{2\mathcal{U}}{\kappa^2\lambda}, \tag{12}$$

gives the expression for q. To obtain this expression in more convenient form we introduce dimensionless variables

$$\Sigma^2 = \frac{\sigma^2}{3H^2}, \tag{13}$$

$$\Omega_\lambda = \frac{\kappa^2\mu^2}{6\lambda H^2}, \tag{14}$$

$$\Omega_\mu = \frac{\kappa^2\mu}{3H^2}, \tag{15}$$

$$\Omega_{\mathcal{U}} = \frac{2\mathcal{U}}{\kappa^2\lambda H^2}. \tag{16}$$

Substituting Eqs.(13)-(16) into (12) we obtain

$$q = 2\Sigma^2 + \frac{1}{2}(3\gamma-2)\Omega_\mu + (3\gamma-1)\Omega_\lambda + \Omega_{\mathcal{U}} \tag{17}$$

and the constraint equation (10) in the form

$$1 = \Sigma^2 + \Omega_\lambda + \Omega_\mu + \Omega_{\mathcal{U}}. \tag{18}$$

With a new time variable τ defined as $\frac{dt}{d\tau} = \frac{1}{H}$ the dynamical equation for the shear takes the form as in standard Bianchi I cosmology [8]

$$(\Sigma^2)' = 2(q-2)\Sigma^2 \tag{19}$$

where $'$ is the derivative with respect to τ. We remind, however, that unlike standard cosmology we can not determine shear components separately unless

$\mathcal{P}_{\mu\nu} = 0$. The Eqs. (17)-(19) take the form obtained in [6] in the particular case $\mathcal{U} = 0$. Using (19) we can easily find that the extremum of the shear parameter Σ corresponds to deceleration parameter $q = 2$. Expressing Σ from (18) and substituting into (17) we find that

$$q = 2 + \frac{3}{2}\Omega_\mu(\gamma - 2) + 3\Omega_\lambda(\gamma - 1) - \Omega_{\mathcal{U}}. \tag{20}$$

This leads to the following equation satisfied at the moment when Σ reaches its extremum:

$$\Omega_{\mathcal{U}} = \frac{3}{2}\Omega_\mu(\gamma - 2) + 3\Omega_\lambda(\gamma - 1). \tag{21}$$

First let us assume $\mathcal{U} = 0$. In this case it is possible to obtaine the result in a very compact form. The equation (21) yields now

$$\Omega_\lambda(\gamma - 1) = -\frac{1}{2}\Omega_\mu(\gamma - 2)$$

Since Ω_λ falls more rapidly than Ω_μ, this extremum of shear is indeed its maximum.

Remembering the definitions of Ω_λ and Ω_μ we can write that $\Omega_\mu/\Omega_\lambda = 2\lambda/\mu$. Using this formula we finally get the ratio of matter density and the brane tension at the moment of maximal shear parameter for a given equation of state:

$$\frac{\mu}{\lambda} = \frac{2 - \gamma}{\gamma - 1}. \tag{22}$$

For the most interesting case of radiation-dominated matter ($\gamma = 4/3$) the maximum of Σ corresponds to $\mu = 2\lambda$.

In the case of the maximally stiff matter ($\gamma = 2$) the isotropisation does not take place even in the standard cosmology, so for γ close to 2 the shear parameter can reach its maximum at an arbitrary late time with an arbitrary small ratio μ/λ. In the opposite case of a very low pressure the isotropisation can begin at a very high matter density. If the pressure is exactly equal to zero ($\gamma = 1$) or it is negative ($\gamma < 1$), the shear parameter decreases from the beginning. In this case there is no maximum of Σ and formula (22) become inapplicable.

If, as often done, we write the equation of state through the parameter w as $p = w\mu$, Eq. (22) takes even more simple and transparent form

$$\frac{\mu}{\lambda} = \frac{1 - w}{w},$$

$w \in (0, 1)$.

In the case $\mathcal{U} \neq 0$ it is necessary to use the full form of (21). It can be rewritten as a quadratic equation for μ/λ

$$\frac{\mu^2}{\lambda^2}(\gamma - 1) + \frac{\mu}{\lambda}(\gamma - 2) = \frac{4\mathcal{U}}{\kappa^4\lambda^2} \tag{23}$$

but with righthandside decreasing with time. For positive $\mathcal{U}$ (otherwise the brane Universe can recollape without reaching the isotropic stage [10]) Eq. (23) has only one positive root, corresponding to maximum of the shear parameter.

As $\mathcal{U}$ decreases like radiation ($\mathcal{U} = C/a^4$), the result obtained for radiation matter without $\mathcal{U}$ can be easily generalised. For radiation-dominated Universe it is possible to write

$$\frac{\mathcal{U}}{\kappa^4 \lambda} = \alpha\mu$$

with some dimensionless constant parameter α. Eq.(23) gives now the matter density at the time of shear maximum as $\mu = \lambda(2+12\alpha)$. In general, for positive $\mathcal{U}$ the isotropisation time shifts to an earlier time than for $\mathcal{U} = 0$. In particular, in the case of the maximally stiff fluid the shear parameter has the maximum at

$$\mu = \frac{2}{\kappa^2}\sqrt{\mathcal{U}}.$$

Clearly, $\mu \to 0$ with $\mathcal{U} \to 0$, reflecting the fact that there is no isotropisation for the maximally stiff fluid without "dark radiation".

We investigated the shear dynamics in Bianchi I cosmology on the brane. Unlike ordinary General Relativity scenario (where the shear parameter monotonically decreases from Kasner initial singularity to isotropic future attractor unless $\gamma = 2$) this dynamics for $\gamma \in (1, 2)$ describes the evolution of the Universe from isotropic singularity to isotropic future attractor through an intermediate anisotropic stage. It was shown that assuming $\sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0$ on the brane it is possible to find the matter density in the epoch of maximum shear parameter analytically. A more restrictive condition $\mathcal{P}_{\mu\nu} = 0$ on the brane allows us to separate the evolution of shear tensor components. An important problem for future development is to find 5-dimensional bulk metric for a Bianchi I braneworld and check the consistency of these $\mathcal{P}_{\mu\nu}$ forms in the bulk.

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